Shinnar- Le Roux RF Pulse Design

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Douze de Abril

OUTLINE

- Pre-Experience:
  - 微方,線代,量物,MR principle and DSP
  - 群論,古力
  - Filter Design
- Bloch Equation & Excitation
  - Review of Bloch Equation & rotating frame
  - From Bloch Equation to Spatial Selective Pulses
  - Solution in matrix representation
- Pulse Design
  - From Bloch Equation to SLR transformation
  - Illustration for SLR pulses design

Section I
Illustration of The Solution to Bloch Equation

微方,線代,古力,量物,

MR Principle

Bloch Equation

$$\frac{d\vec{M}(t)}{dt} + \frac{\vec{M}(t) - \chi\vec{B}(t)}{T} + \gamma(\vec{B}(t) \times \vec{M}(t)) = 0$$

- Bloch Eq in vector form
- Solve this equation, and you will find what you want

γ : gyro-magnetic ratio

H : magnetic field

M : magnetic moment

T : relaxation parameter

Bloch Equation in Matric Representation

$$\frac{d}{dt}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{T_2} & 0 & 0 \\ 0 & \frac{1}{T_2} & 0 \\ 0 & 0 & \frac{1}{T_1} \end{bmatrix}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} + \gamma(\begin{bmatrix} 0 & -B_z(t) & B_y(t) \\ B_z(t) & 0 & -B_x(t) \\ -B_y(t) & B_x(t) & 0 \end{bmatrix}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix}) = \chi\begin{bmatrix} \frac{B_x(t)}{T_2} \\ \frac{B_y(t)}{T_2} \\ \frac{B_z(t)}{T_1} \end{bmatrix}$$

磁矩變化+磁矩遲緩瞬變率+磁場引起的磁矩瞬變量

=外加磁場遲緩瞬變率（本日最難算）

A Linear Coupling Dynamic System

Properties

- A Linear Dynamic System

$$\frac{d}{dt}\vec{M}(t) + \widetilde{B}(t)\vec{M}(t) = \vec{F}(t)$$

- Cannot be reduced into a linear equation
- Linear Response Theorem may not be applied. (Fourier: C'est un grand probléme)
- Numerical Method

  (Le Roux: Ce n'est pas grave. J'ai un ordinateur)

Simplify The Equation

- Relaxation Time 10~ 100 ms
  && RF duration ~1ms
- Drop the relaxation part
- Homogeneous Differential Equation.

$$\frac{d}{dt}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} + \gamma(\begin{bmatrix} 0 & -B_z(t) & B_y(t) \\ B_z(t) & 0 & -B_x(t) \\ -B_y(t) & B_x(t) & 0 \end{bmatrix}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix}) = 0$$

It can’t be simplified anymore

Equation of Linear operators

$$\frac{d}{dt}\vec{M}(t)+\widetilde{B}(t)\vec{M}(t)=0$$

$$\vec{M}(t)=Exp(-\widetilde{B}(t))\cdot\vec{M}(0)$$

↑

What’s this???

Warming Up Exercise

- We put a magnetic dipole moment M in a (0,0,B) homogenous Magnet.

$$\frac{d}{dt}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} + \gamma\left(\begin{bmatrix} 0 & -B & 0 \\ B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix}\right) = 0$$

$$\frac{d}{dt} M_x(t) - B\gamma M_y(t) = 0$$

$$\frac{d}{dt} M_y(t) + B\gamma M_x(t) = 0$$

$$\frac{d}{dt} M_z(t) = 0 \Leftrightarrow M_z(t) = const$$

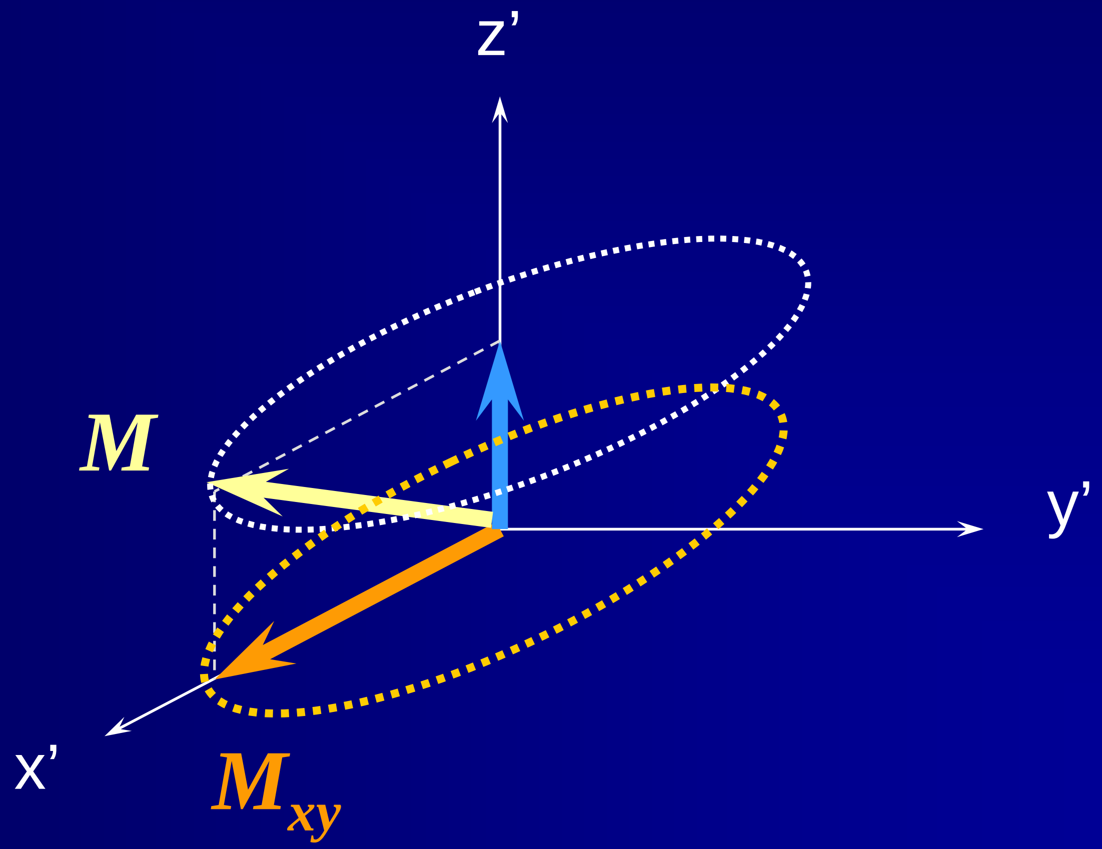
M is rotating around z axis

$$\begin{cases} \frac{d}{dt}M_x(t) - B\gamma M_y(t) = 0 \Leftrightarrow \frac{d^2}{dt^2}M_x(t) - B\gamma\frac{d}{dt}M_y(t) = 0 \\ \frac{d}{dt}M_y(t) + B\gamma M_x(t) = 0 \Leftrightarrow \frac{d^2}{dt^2}M_y(t) + B\gamma\frac{d}{dt}M_x(t) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{d^2}{dt^2}M_x(t) + (B\gamma)^2 M_x(t) = 0 \Leftrightarrow M_x(t) = C_1 e^{jB\gamma t} + C_2 e^{-jB\gamma t} \\ \frac{d^2}{dt^2}M_y(t) + (B\gamma)^2 M_y(t) = 0 \Leftrightarrow M_y(t) = C_3 e^{jB\gamma t} + C_4 e^{-jB\gamma t} \end{cases}$$

$$\Leftrightarrow \begin{cases} M_x(t) = C_1 e^{jB\gamma t} + C_2 e^{-jB\gamma t} \\ M_y(t) = i[C_1 e^{jB\gamma t} - C_2 e^{-jB\gamma t}] = C_1 e^{j(B\gamma t + \frac{\pi}{2})} + C_2 e^{-j(B\gamma t \frac{\pi}{2})} \end{cases}$$

Illustration of Spin Precession



Brief Summary

- Constant Magnetic Field

  $\Rightarrow$Precession in Larmor frequency

- Solution of Simple Version of Bloch EQ.

$$\vec{M}(t) = Exp(-\widetilde{B}(t)) \cdot \vec{M}(0)$$

It seems to
be a rotation

Section II
Excitation, Rotating Frame & Selective Excitation

MR principle

Principal Reference : 鍾老師MR principle
第一堂課的assigned reading

Pushing Over A Magnetic Moment

- Classical Description
  - M rotates in high speed under a magnetic field

1. Catching M before pushing it down
2. Catcher: Fast Oscillating EM Field
3. Pusher: A relative static magnetic field B1
4. Excitation Complete $\theta = \gamma \boldsymbol{B_1} \tau$

Selective Excitation

1. Apply A Spatial Magnetic Gradient
2. Accordingly, local precession frequency is changed
3. Send out a RF pulse with corresponding bandwidth to push over the magnetic moment.

- It seems too simple and too perfect. How could this work???
- I have to apologize for any confusion between slides and readings
  - Sagittal Slice Selection
  - To keep away from confusing notation

Recalling Bloch Equation & Rotating Frame

$$\left[\frac{d\vec{M}(t)}{dt}\right]_{fix} + \gamma\vec{B}(t)\times\vec{M} = 0$$

$$\left[\frac{d\vec{M}(t)}{dt}\right]_{fix} = \left[\frac{d\vec{M}(t)}{dt}\right]_{rot} + \vec{\omega}\times\vec{M}$$

$$\left[\frac{d\vec{M}(t)}{dt}\right]_{rot} = -(\gamma\vec{B}(t)\times\vec{M} + \vec{\omega}_r\times\vec{M})$$

Choose ω to eliminate static magnetic field in rotating frame

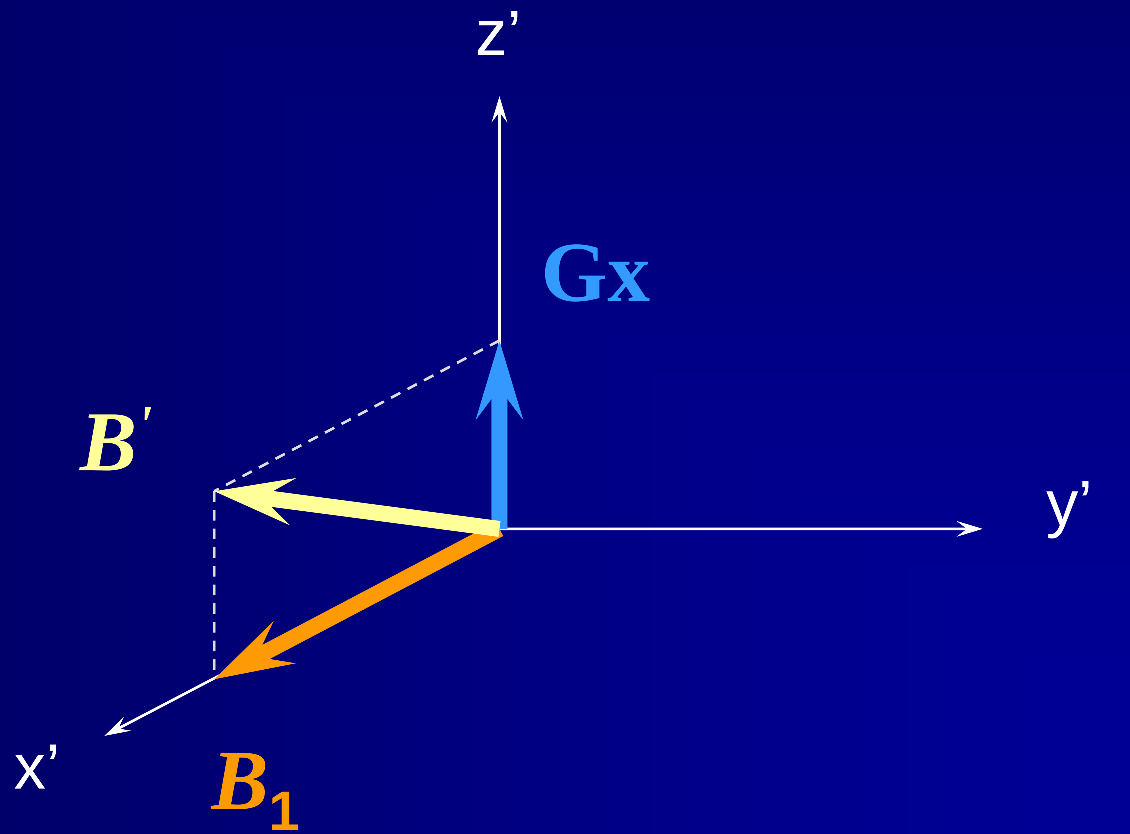
Slice Selective Excitation in Rotating Frame

$$\left[\frac{d\vec{M}(t)}{dt}\right]_{rot} = -(\gamma\vec{B}_0(t)\times\vec{M} + \vec{\omega}_r \times \vec{M} + \gamma\vec{B}_1(t)\times\vec{M} + \gamma\vec{B}_{grad}(x,t)\times\vec{M})$$

$$= -\gamma(\vec{B}_1(t) + \vec{B}_{grad}(x,t))\times\vec{M}$$

- Recall that this is a dynamic equation for rotation
- The rotating angular velocity is $\gamma\left|\vec{B}_1(t)+\vec{B}_{grad}(x,t)\right|$
- The rotation axis is the direction of the vector sum of magnetic field

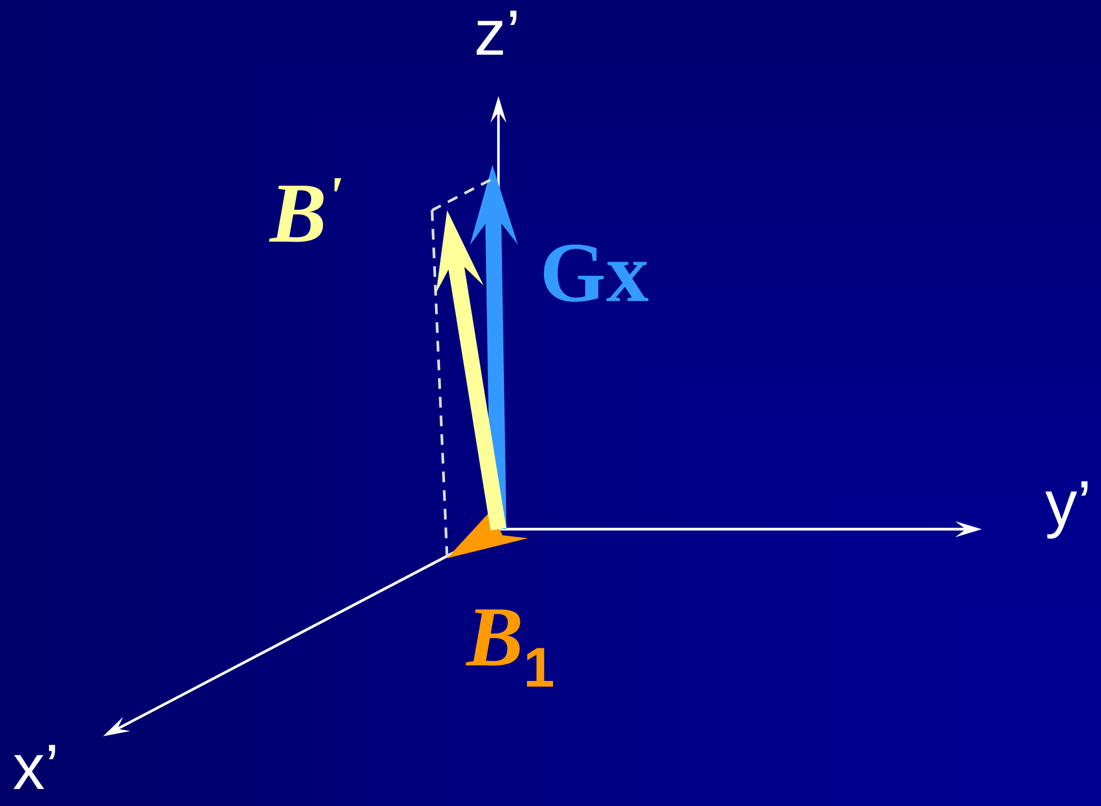
Illustration for Slice Selective Excitation



Example

- A 90 degree excitation for a 0.5 cm slice is required in 1.5 T system within 2ms. The maximum gradient the system would achieve is 1.0G/cm.
- B1=0.03Gauss
- ΔB(x)=+0.25 Gauss ~ -0.25 Gauss

Illustration for Slice Selective Excitation



Pi/2 Excitation Profile

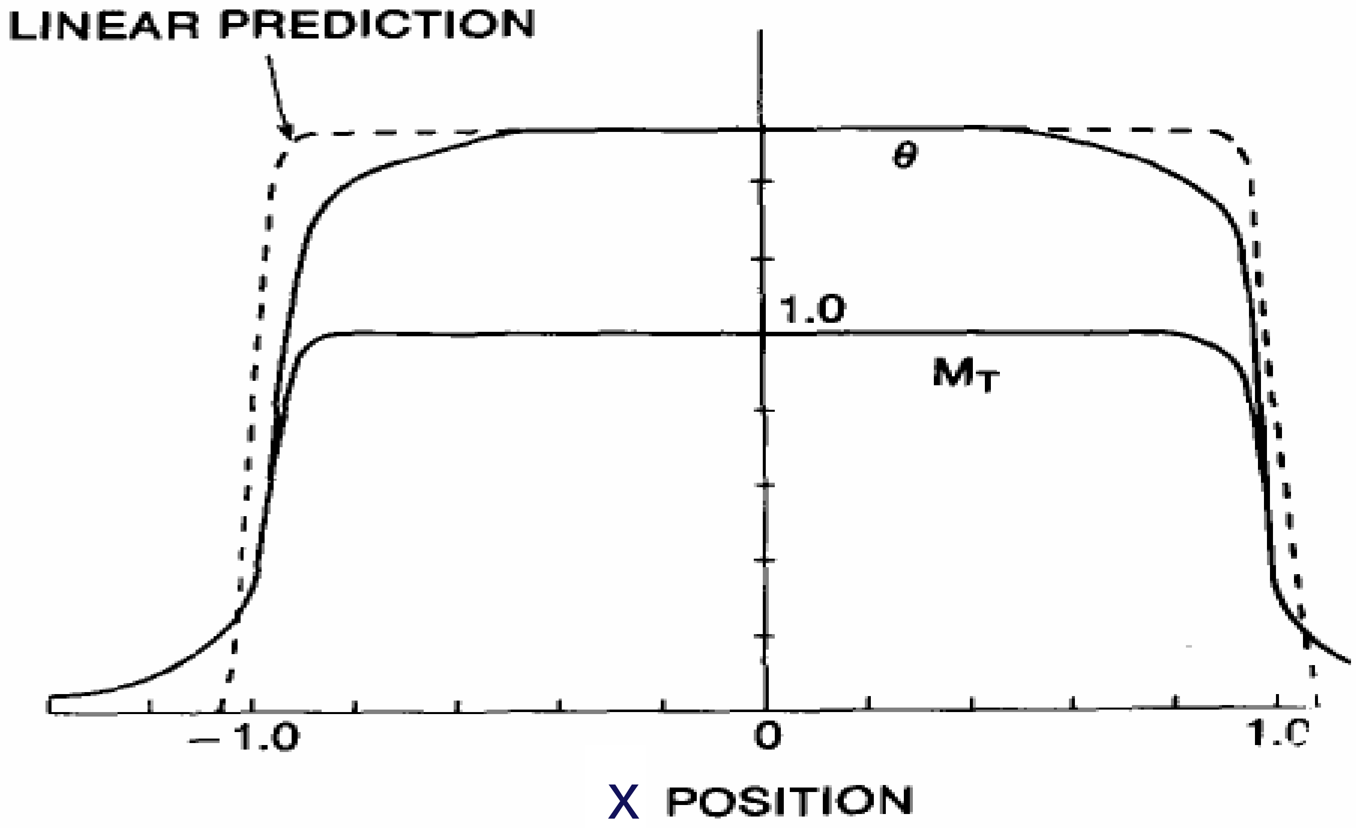


FIG. 1. Flip angle (θ) and M_T vs z position for a 90° sinc pulse.

Pi Excitation Profile

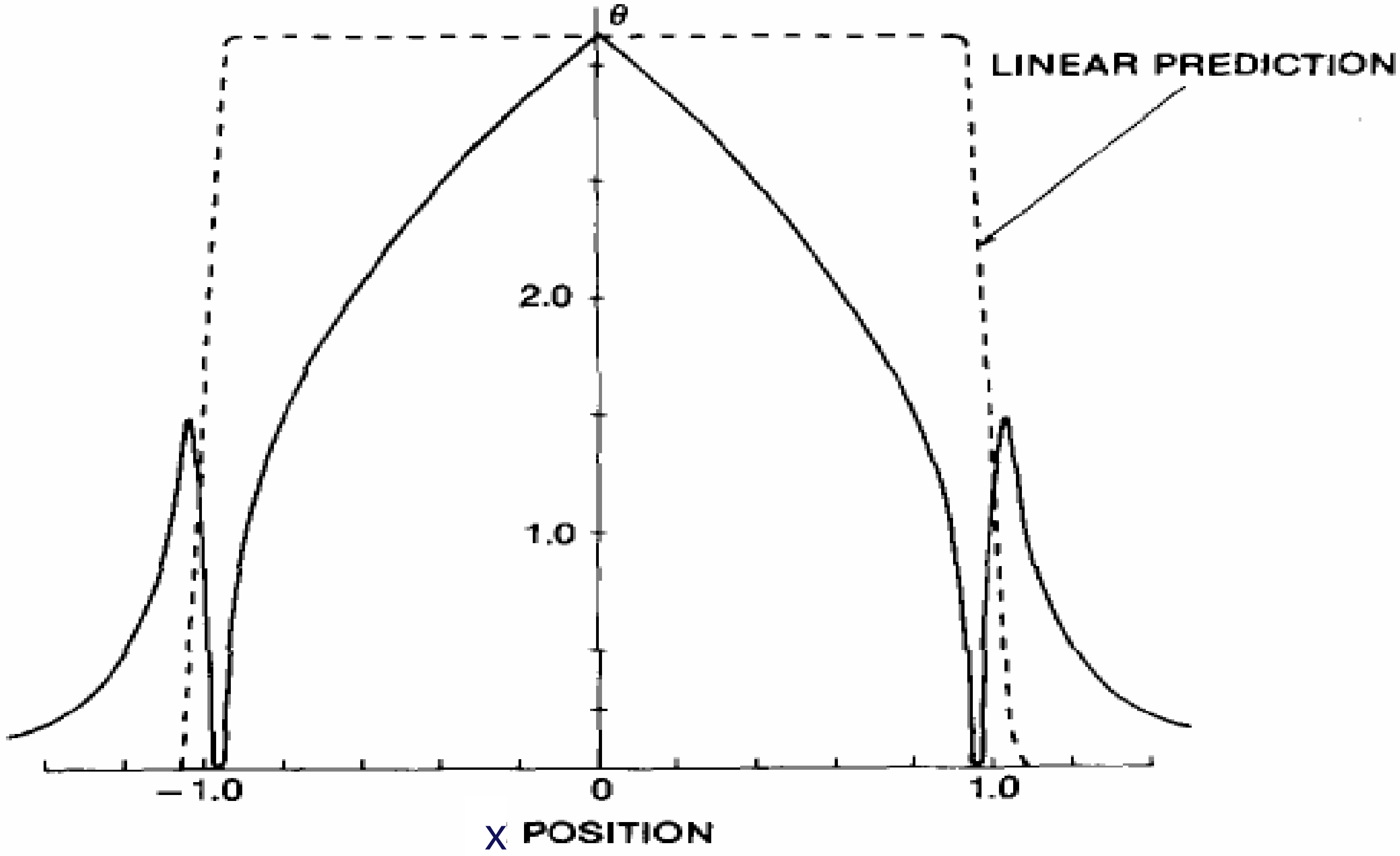


FIG. 2. Flip angle (θ) for a 180° sinc pulse.

Brief Summary for Sec I and II

- Bloch equation regardless of relaxation : A dynamic rotation equation

  Axis : The direction of net magnetic field

  angular frequency : $\gamma \boldsymbol{B}$
- Sinc RF would achieve a small angle selective excitation or any flip angles under less off-resonance circumstances.

Imaging ???

- 影像咧~~~
  - ”Pi什麼Pi, 90度都不90度了” by Mr. Hu
  - Spin Echo ? 哇~~~Echo怎麼這麼小
    預計看到5mm的slice結果只有3mm
  - FSE, 怎麼突然好像出現一堆Stimulated Echo

- 不准偷懶，
  硬著頭皮解Bloch Equation吧

FSE Images

SLR Sinc

Single Slice

Multi-Slice

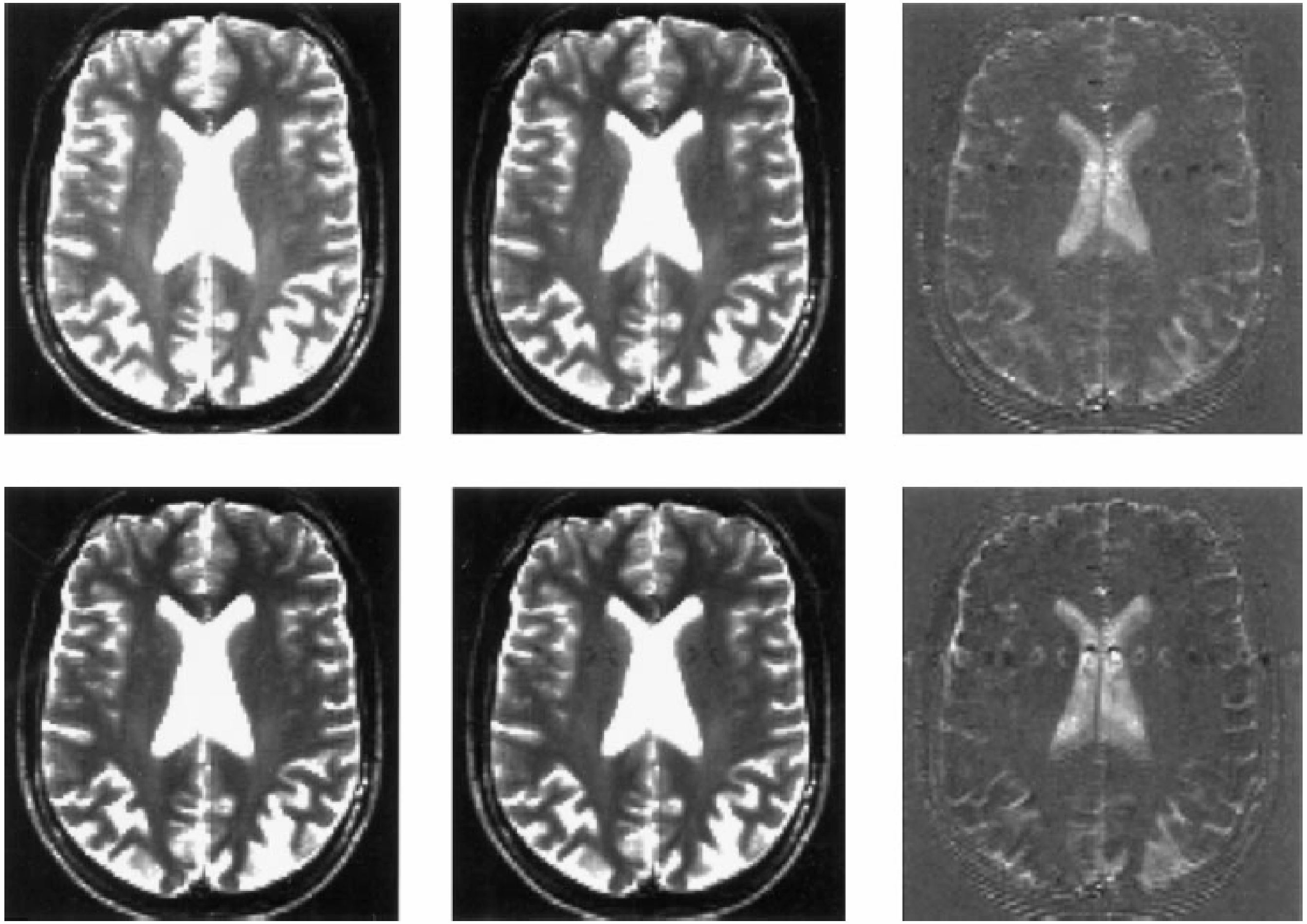


Figure 7. Volunteer study. Comparison between *slr1*-FSE (left) and *sie*-FSE (mid) images. They were obtained by using one-slice (top) and seven-slice (bottom) measurements. In the right column the difference between *slr1* and *sie* images is shown.

Section III
Introduction to Matrix Manipulation On Spins

線代+群論, 古力

Gleam!!!

- Magnetic field ↔ Rotating a magnetic moment (A simple description)
- Rotating ↔ Rotating Matrix ↔ SO(3) 3D rotation ↔ SU(2) simple version

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longleftrightarrow \begin{bmatrix} z & x - jy \\ x + jy & -z \end{bmatrix}$$

- Rotation becomes Spatial Transformation
- Let's Roll the magnetic moments.

Rotation About z axis

Special Orthogonal group(3) Representation

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Special Unitary group(2) Representation

$$\begin{bmatrix} e^{-\frac{j\theta}{2}} & 0 \\ 0 & e^{\frac{j\theta}{2}} \end{bmatrix}$$

Rotation About x axis

Special Orthogonal group(3) Representation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Special Unitary group(2) Representation

$$\begin{bmatrix} \cos\frac{\theta}{2} & -j\sin\frac{\theta}{2} \\ -j\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

Rotation About y axis

Special Orthogonal group(3) Representation

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Special Unitary group(2) Representation

$$\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

Spin domain representation

$$\begin{cases} a_i = \cos\frac{\phi_i}{2} + j\hat{n}_{z,i}\sin\frac{\phi_i}{2} \\ b_i = j(\hat{n}_{x,i} + j\hat{n}_{y,i})\sin\frac{\phi_i}{2} \end{cases}$$

$$\begin{cases} \phi_i = \gamma\Delta t\sqrt{\left|\vec{B}_{1,i}\right|^2 + (Gx)^2} \\ \hat{n}_i = \frac{\gamma\Delta t}{\left|\phi_i\right|}(\vec{B}_{1x,i}, \vec{B}_{1y,i}, Gx) \end{cases}$$

$$Q_i = \begin{bmatrix} a_i & -b_i^* \\ b_i & a_i^* \end{bmatrix}$$

$\vec{B}_{1,i}$: *the i − th magnetic field*

Δt : *the turned − on duration of* $\vec{B}_{1,i}$

G : *the gradient strength*

x : *the position*

Spin Rotation

$$Q_i = \begin{bmatrix} a_i & -b_i^* \\ b_i & a_i^* \end{bmatrix}$$

A Rotation about an arbitrary axis

$$Det(Q_i) = 1$$

A Series of rotation operation

$$\begin{bmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix} = \begin{bmatrix} a_i & -b_i^* \\ b_i & a_i^* \end{bmatrix} \begin{bmatrix} a_{i-1} & -b_{i-1}^* \\ b_{i-1} & a_{i-1}^* \end{bmatrix} \cdots \begin{bmatrix} a_1 & -b_1^* \\ b_1 & a_1^* \end{bmatrix} \begin{bmatrix} a_0 & -b_0^* \\ b_0 & a_0^* \end{bmatrix}$$

a, b代表每個被分開的小轉動

α , β 代表一堆a,b和程的大轉動

Spin Rotation

$$\begin{bmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix}$$

$$\begin{cases} \alpha_i = \cos\dfrac{\Phi_i}{2} + j\widehat{N}_{z,i}\sin\dfrac{\Phi_i}{2} \\ \beta_i = j(\widehat{N}_{x,i} + j\widehat{N}_{y,i})\sin\dfrac{\Phi_i}{2} \end{cases}$$

Φ_i : *the effective flip – angle for the combination of rotations*

$\widehat{N}$: *the effective rotating – axis for the combination of rotations*

偷懶新招式

$$\begin{bmatrix}\alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^*\end{bmatrix} = \begin{bmatrix}a_i & -b_i^* \\ b_i & a_i^*\end{bmatrix}\begin{bmatrix}a_{i-1} & -b_{i-1}^* \\ b_{i-1} & a_{i-1}^*\end{bmatrix}\cdots\begin{bmatrix}a_1 & -b_1^* \\ b_1 & a_1^*\end{bmatrix}\begin{bmatrix}a_0 & -b_0^* \\ b_0 & a_0^*\end{bmatrix}$$

No rotation as the Initial Condition

$$\begin{bmatrix}\alpha_1 \\ \beta_1\end{bmatrix} = \begin{bmatrix}\alpha_1 & -\beta_1^* \\ \beta_1 & \alpha_1^*\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

The Product of two unitary matrices is also unitary

$$\begin{bmatrix}\alpha_i \\ \beta_i\end{bmatrix} = \begin{bmatrix}a_i & -b_i^* \\ b_i & a_i^*\end{bmatrix}\begin{bmatrix}\alpha_{i-1} \\ \beta_{i-1}\end{bmatrix}$$

Spin Rotation

$$M_{xy} = M_x + jM_y$$

$$\begin{bmatrix} M_{xy}(t_i) \\ M_{xy}^*(t_i) \\ M_z(t_i) \end{bmatrix} = \begin{bmatrix} (a_i^*)^2 & -(b_i)^2 & 2a_i^* b_i \\ -(b_i^*)^2 & (a_i)^2 & 2a_i b_i^* \\ -a_i^* b_i^* & -a_i b_i & a_i a_i^* - b_i b_i^* \end{bmatrix} \begin{bmatrix} M_{xy}(t_{i-1}) \\ M_{xy}^*(t_{i-1}) \\ M_z(t_{i-1}) \end{bmatrix}$$

$$\begin{bmatrix} M_{xy}(t_i) \\ M_{xy}^*(t_i) \\ M_z(t_i) \end{bmatrix} = \begin{bmatrix} (\alpha_i^*)^2 & -(\beta_i)^2 & 2\alpha_i^* \beta_i \\ -(\beta_i^*)^2 & (\alpha_i)^2 & 2\alpha_i \beta_i^* \\ -\alpha_i^* \beta_i^* & -\alpha_i \beta_i & \alpha_i \alpha_i^* - \beta_i \beta_i^* \end{bmatrix} \begin{bmatrix} M_{xy}(0) \\ M_{xy}^*(0) \\ M_z(0) \end{bmatrix}$$

A simple form

$$\begin{bmatrix} M_z(t_i) & M_x(t_i) - j\cdot M_x(t_i) \\ M_x(t_i) + j\cdot M_x(t_i) & -M_z(t_i) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix} \begin{bmatrix} M_z(0) & M_x(0) - j\cdot M_x(0) \\ M_x(0) + j\cdot M_x(0) & -M_z(0) \end{bmatrix} \begin{bmatrix} \alpha_i^* & \beta_i^* \\ -\beta_i & \alpha_i \end{bmatrix}$$

么正矩陣的共軛轉秩矩陣爲
其反矩陣

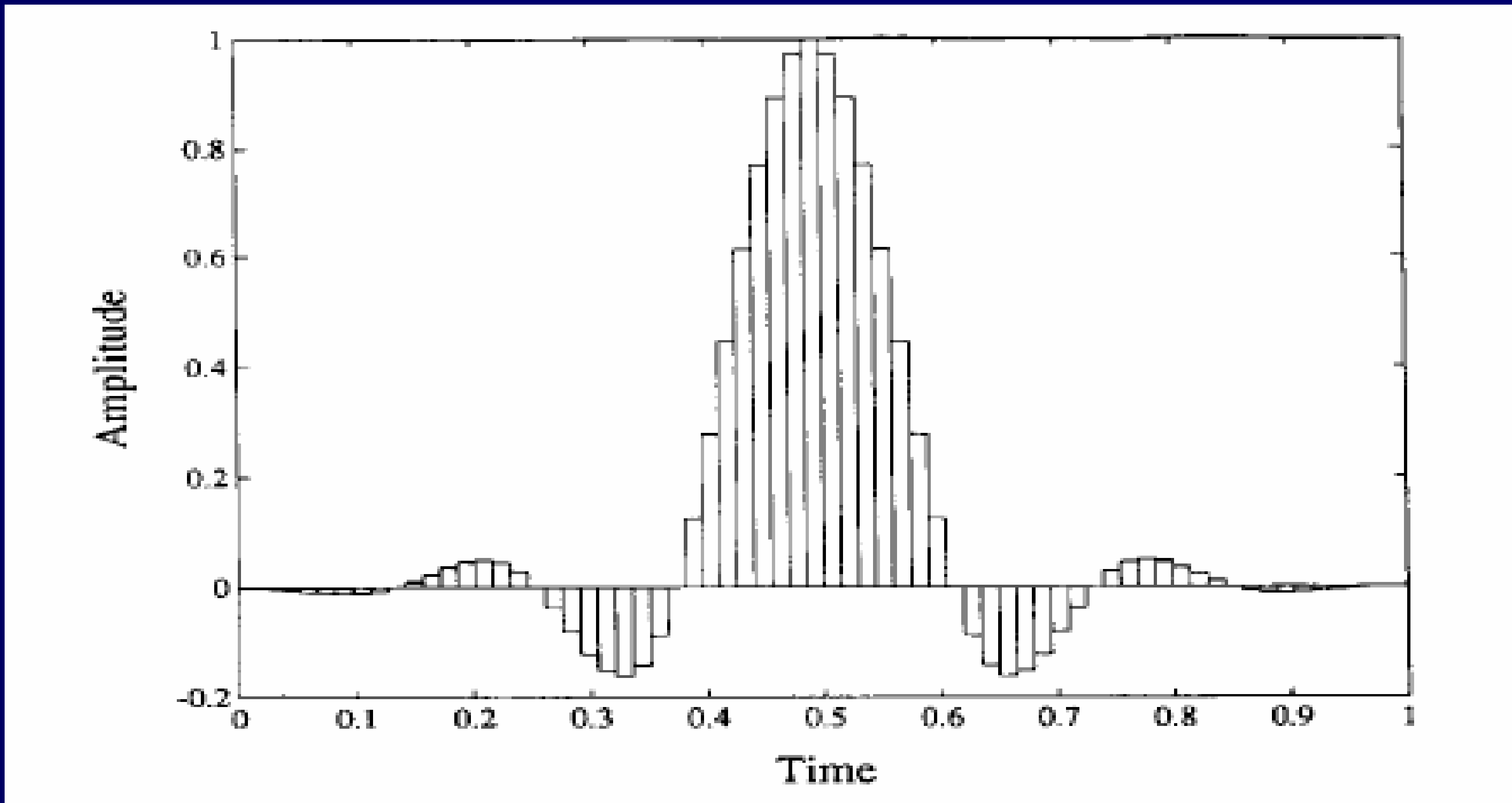
Brief Summary of Sec III

- Cayley-Klein rotation would simplify the analysis in RF excitation problem
- An entire excitation could be regarded as a series of excitations.
- However, what about time-varying magnetic fields??

Section IV
Shinnar – Le Roux Transformaiton

線代, DSP

Piece-Wise Constant Approximation



Hard Pulse Approximation

- Differentiation first, then Integration
- Differentiation => Infinitesimal Rotation
- But, there is not only B_1 but Gradient
- A combination of small rotation => Sequential Rotation

Selective RF Pulse in Hard Pulse Approximation

$$Q_i = \begin{bmatrix} a_i & -b_i^* \\ b_i & a_i^* \end{bmatrix} \qquad \begin{cases} a_i = \cos\dfrac{\phi_i}{2} + j\hat{n}_{z,i}\sin\dfrac{\phi_i}{2} \\ b_i = j(\hat{n}_{x,i} + j\hat{n}_{y,i})\sin\dfrac{\phi_i}{2} \end{cases} \qquad \begin{cases} \phi_i = \gamma\Delta t\sqrt{\left|\vec{B}_{1,i}\right|^2 + (Gx)^2} \\ \hat{n}_i = \dfrac{\gamma\Delta t}{\left|\phi_i\right|}(\vec{B}_{1x,i}, \vec{B}_{1y,i}, Gx) \end{cases}$$

HARD PULSE APPROXIMATION

Separate the rotation into RF part and Gradient Part

$$Q_i = \begin{bmatrix} C_i & -S_i^* \\ S_i & C_i \end{bmatrix} \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix} \qquad \begin{cases} C_i = \cos(\gamma\left|\vec{B}_{1,i}\right|\Delta t/2) \\ S_i = je^{j\angle\vec{B}_{1,i}}\sin(\gamma\left|\vec{B}_{1,i}\right|\Delta t/2) \\ z = e^{jGx\Delta t} \end{cases}$$

C是實數所以共軛符號可省略

Forward SLR

Selective RF Excitation

$$Q_i = \begin{bmatrix} C_i & -S_i^* \\ S_i & C_i \end{bmatrix} \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix}$$

Spin State Representation

$$\begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} = Q_i \begin{bmatrix} \alpha_{i-1} \\ \beta_{i-1} \end{bmatrix} = \begin{bmatrix} C_i & -S_i^* \\ S_i & C_i \end{bmatrix} \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix} \begin{bmatrix} \alpha_{i-1} \\ \beta_{i-1} \end{bmatrix}$$

Forward SLR

$$\begin{bmatrix}\alpha_i\\ \beta_i\end{bmatrix} = z^{1/2}\begin{bmatrix}C_i & -S_i^*\\ S_i & C_i\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}\begin{bmatrix}\alpha_{i-1}\\ \beta_{i-1}\end{bmatrix}$$

代入

$$Define: \begin{bmatrix}A_i\\ B_i\end{bmatrix} = z^{\frac{i}{2}}\begin{bmatrix}\alpha_i\\ \beta_i\end{bmatrix}$$

$$\begin{bmatrix}A_i\\ B_i\end{bmatrix} = \begin{bmatrix}C_i & -S_i^*\\ S_i & C_i\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}\begin{bmatrix}A_{i-1}\\ B_{i-1}\end{bmatrix}$$

Forward SLR

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} C_1 & -S_1^* \\ S_1 & C_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 & -S_1^* \\ S_1 & C_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} C_2 & -S_2^* z^{-1} \\ S_2 & C_2 z^{-1} \end{bmatrix} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} = \begin{bmatrix} C_1 C_2 - S_2^* S_1 z^{-1} \\ S_2 C_1 + C_2 S_1 z^{-1} \end{bmatrix}$$

$$A_n(z) = \sum_{i=0}^{n-1} a_i z^{-i}$$

$$B_n(z) = \sum_{i=0}^{n-1} b_i z^{-i}$$

失落的環節

不好意思
突然想到這樣表達會比較好

α_n, β_n才是真正的轉動矩陣元素

$$\begin{bmatrix} M_z(x,t_i) & M_x(x,t_i) - j\cdot M_x(x,t_i) \\ M_x(x,t_i) + j\cdot M_x(x,t_i) & -M_z(x,t_i) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_i(x,t_i) & -\beta_i^*(x,t_i) \\ \beta_i(x,t_i) & \alpha_i^*(x,t_i) \end{bmatrix} \begin{bmatrix} M_z(x,0) & M_x(x,0) - j\cdot M_x(x,0) \\ M_x(x,0) + j\cdot M_x(x,0) & -M_z(x,0) \end{bmatrix} \begin{bmatrix} \alpha_i^*(x,t_i) & \beta_i^*(x,t_i) \\ -\beta_i(x,t_i) & \alpha_i(x,t_i) \end{bmatrix}$$

$$Define: \begin{bmatrix} A_i \\ B_i \end{bmatrix} = z^{\frac{i}{2}} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \Leftrightarrow z^{-\frac{i}{2}} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$$

再把A_n, B_n換成α_n, β_n

$$A_n(z) = \sum_{i=0}^{n-1} a_i z^{-i} \qquad \Leftrightarrow \qquad \alpha_n(z) = z^{-\frac{n}{2}}\left(\sum_{i=0}^{n-1} a_i z^{-i}\right)$$

$$B_n(z) = \sum_{i=0}^{n-1} b_i z^{-i} \qquad\qquad \beta_n(z) = z^{-\frac{n}{2}}\left(\sum_{i=0}^{n-1} b_i z^{-i}\right)$$

再想一想α_n, β_n實際的意義

$$\begin{cases} \alpha_n = \cos\dfrac{\Phi_n}{2} + j\widehat{N}_{z,i}\sin\dfrac{\Phi_n}{2} \\ \beta_n = j(\widehat{N}_{x,i} + j\widehat{N}_{y,i})\sin\dfrac{\Phi_n}{2} \end{cases}$$

把Z多項式換成與X相關的函數

$$\alpha_n(z \to e^{-jGx\Delta t}) = z^{-\frac{n}{2}}\left(\sum_{i=0}^{n-1} a_i z^{-i}\right)$$

$$\beta_n(z \to e^{-jGx\Delta t}) = z^{-\frac{n}{2}}\left(\sum_{i=0}^{n-1} b_i z^{-i}\right)$$

$$\begin{cases} \alpha_n = \cos\frac{\Phi_n(x)}{2} + j\hat{N}_{z,i}\sin\frac{\Phi_n(x)}{2} \\ \beta_n = j(\hat{N}_{x,i}(x) + j\hat{N}_{y,i}(x))\sin\frac{\Phi_n(x)}{2} \end{cases}$$

括號中的x代表位置，

下標的x代表B_1的x方向

- 把 α , β 展開後個別分出實部和虛部
- 不論是實部或虛部都是x的函數
- 帶入右邊的關係，就可以得到每一個位置的effective flip-angle和phase

Forward SLR

- A Given RF profile combining a spatial gradient
- Dividing those profiles into piece-wise square function
- Find corresponding state space rotation matrix
- Forward SLR would give the slice excitation profile
- An and B_n are polynomials of z of the order of -(n-1)

Inverse SLR

- Generally, a specific excitation profile is desired
- Deduce the rotation matrices from the slice profile demanded to the initial state
- Inverse Matrix Manipulation: A profit from matrix representation

Inverse SLR

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} C_i & -S_i^* \\ S_i & C_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} A_{i-1} \\ B_{i-1} \end{bmatrix}$$

Inverse Matrix Operation

$$\begin{bmatrix} A_{i-1} \\ B_{i-1} \end{bmatrix} = \left(\begin{bmatrix} C_i & -S_i^* \\ S_i & C_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & z^{1} \end{bmatrix} \begin{bmatrix} C_i & S_i^* \\ -S_i & C_i \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

Inverse SLR

- Don’t Forget A_n and B_n are polynomials of z of the order –(n-1)
- The highest term of A_{i-1} should be 0
- The lowest term of B_{i-1} should be 0

$$\begin{bmatrix} A_{i-1} \\ B_{i-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z^{1} \end{bmatrix} \begin{bmatrix} C_i & S_i^* \\ -S_i & C_i \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} C_i A_i + S_i^* B_i \\ z(-S_i A_i + C_i B_i) \end{bmatrix}$$

Inverse SLR

$$C_i A_{i,i-1} + S_i^* B_{i,i-1} = 0$$

$$-S_i A_{i,0} + C_i B_{i,0} = 0$$

$A_{i,m}$: *the mth term of the polynomial* A_i

$B_{i,m}$: *the mth term of the polynomial* B_i

$$\begin{cases} C_i = \cos(\gamma |\vec{B}_{1,i}| \Delta t / 2) \\ S_i = j e^{j\angle \vec{B}_{1,i}} \sin(\gamma |\vec{B}_{1,i}| \Delta t / 2) \\ z = e^{jGx\Delta t} \end{cases}$$

$$\frac{B_{i,0}}{A_{i,0}} = \frac{S_i}{C_i} = \frac{j e^{j\angle \vec{B}_{1,i}} \sin(\gamma |\vec{B}_{1,i}| \Delta t / 2)}{\cos(\gamma |\vec{B}_{1,i}| \Delta t / 2)}$$

Inverse SLR

$$\frac{B_{i,0}}{A_{i,0}} = \frac{je^{j\angle\vec{B}_{1,i}} \sin(\gamma\left|\vec{B}_{1,i}\right|\Delta t/2)}{\cos(\gamma\left|\vec{B}_{1,i}\right|\Delta t/2)} = \frac{je^{j\theta_i} \sin(\phi_i/2)}{\cos(\phi_i/2)}$$

$$\tan(-\gamma\left|\vec{B}_{1,i}\right|\Delta t/2) = \left|\frac{B_{i,0}}{A_{i,0}}\right| = \tan(\phi_i/2) \longrightarrow$$

Flip Angle resulting from the i-th piece-wise magnetic field

$$\theta_i = phase(-j\frac{B_{i,0}}{A_{i,0}}) \longrightarrow$$

Phase Angle resulting from the i-th piece-wise magnetic field

Inverse SLR

The RF pulse Profile

$$\vec{B}_{1,i} = \frac{1}{\gamma \Delta t} \phi_i e^{j\theta_i}$$

Forward SLR怎麼求得Slice 的 Flip-Angle和phase

Inverse SLR就依循相同的方式反推

Inverse SLR Example

$$\alpha_n(z \rightarrow e^{-jGx\Delta t}) = z^{-\frac{n}{2}}\left(\sum_{i=0}^{n-1} a_i z^{-i}\right)$$

這部分是$A_n(z)$

$$\beta_n(z \rightarrow e^{-jGx\Delta t}) = z^{-\frac{n}{2}}\left(\sum_{i=0}^{n-1} b_i z^{-i}\right)$$

這個部分是一個Gradient造成的Linear Phase

這部分是$B_n(z)$

Inverse SLR Example

- 設計一個RF pulse
- 這個RF pulse只有rotating frame的x’方向
- 只要激發一個slice，
- 激發完後每個地方的Flip Angle必須要是 $\Phi(x)$
- 整個Pulse 的Total Phase由$B_n(z)$的phase與Gradient 造成的phase共同決定

Inverse SLR Example

- Design the excitation profile
- Use the polynomaial $B_n(z)$ to approximate the profile
- Calculate $A_n(z)$ by constraints
- Find Magnetic Pulse

$$B_n(Z \rightarrow e^{j\gamma Gx\Delta t})$$

$$= j(\widehat{N}_{x,n} + j\widehat{N}_{y,n})\sin\frac{\Phi(x)}{2}$$

$$= j\sin\frac{\Phi(x)}{2}$$

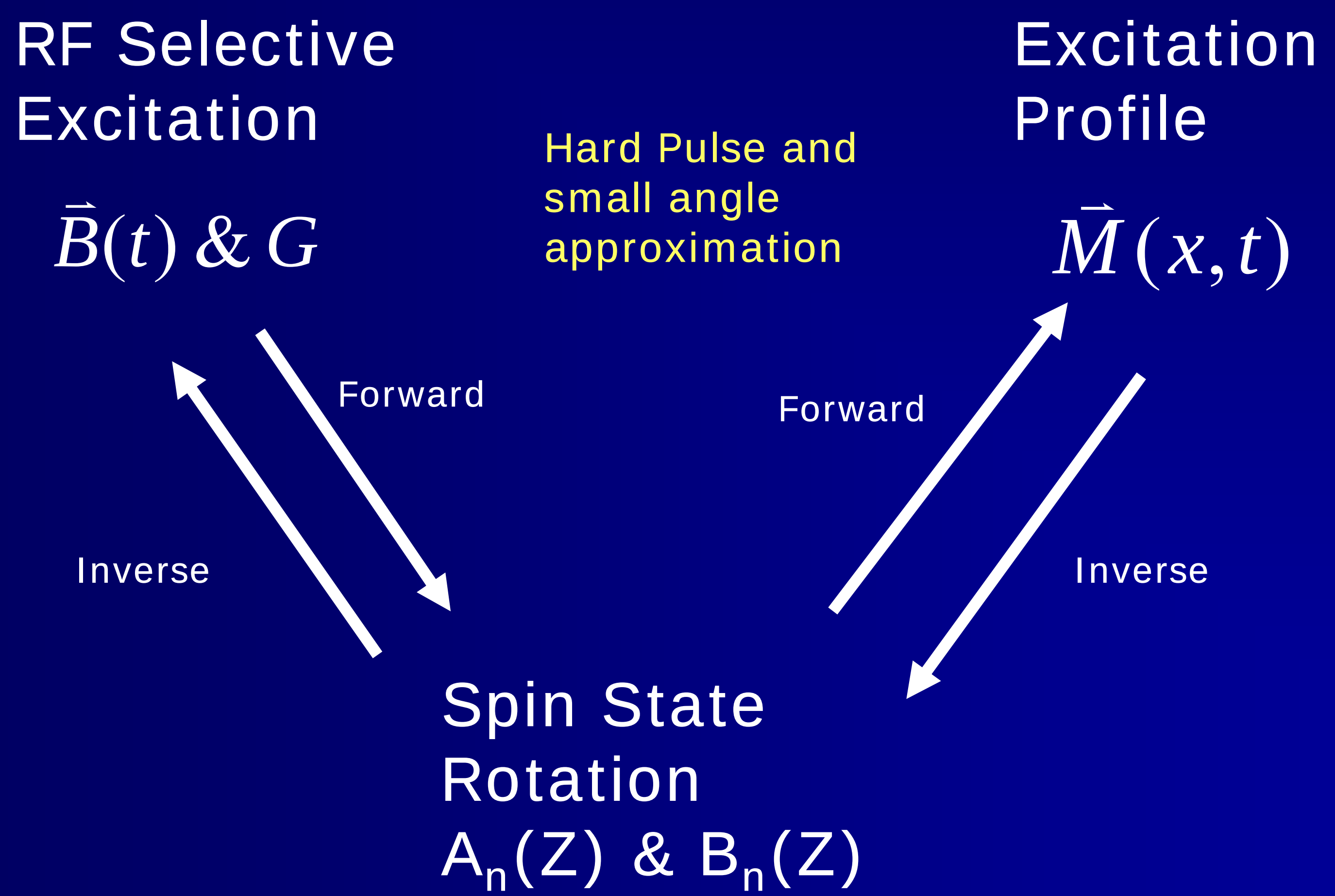
$$|A_n(Z)| = \sqrt{1 - |B_n(Z)|^2}$$

$$\vec{B}_{1,i} = \frac{1}{\gamma\Delta t}\phi_j e^{j\theta_i}$$

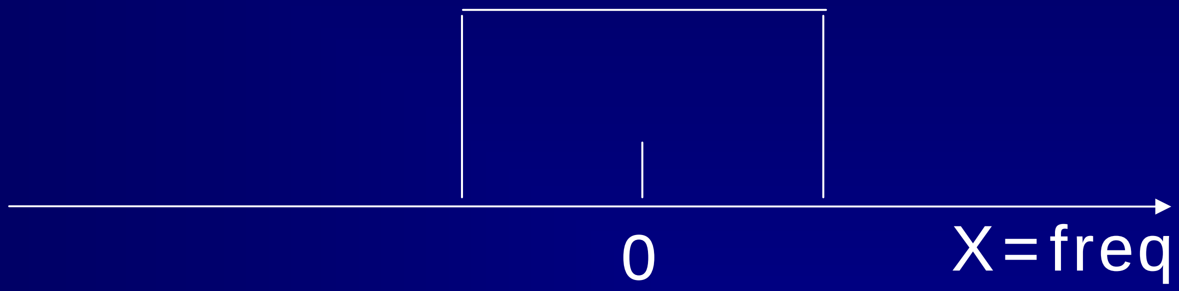
Brief Summary

- SLR transform :
  - A more exact approximation of the solution to the Bloch Equation.
- Forward SLR transform
  - Given A Magnetic Field
  - Calculate the corresponding excitation profile
- Inverse SLR transform
  - Given a desired excitation profile
  - Find the appropriate rotation sequence
  - Calculate corresponding B field

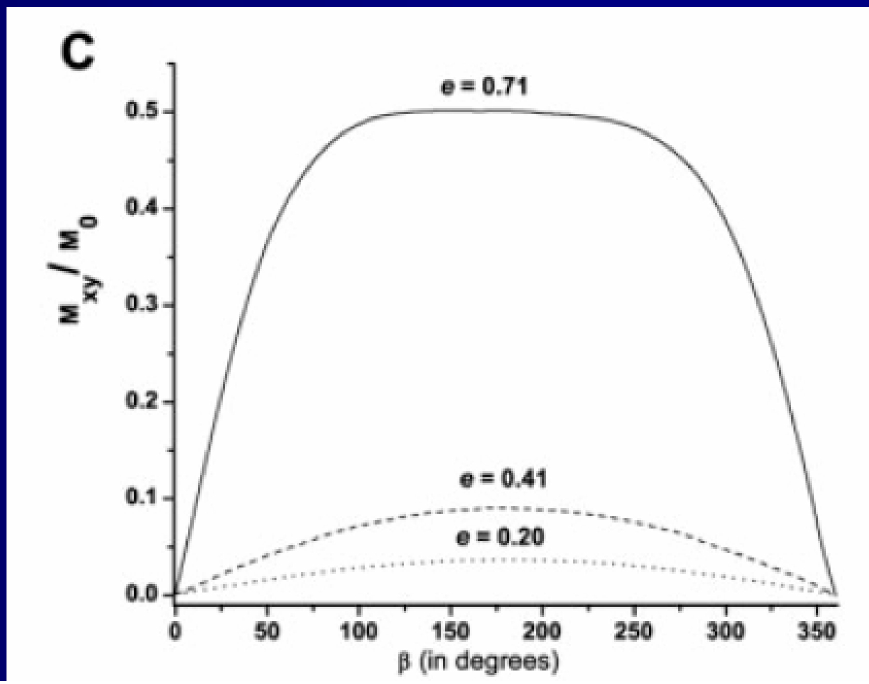
SLR Algorithm



Interesting Application

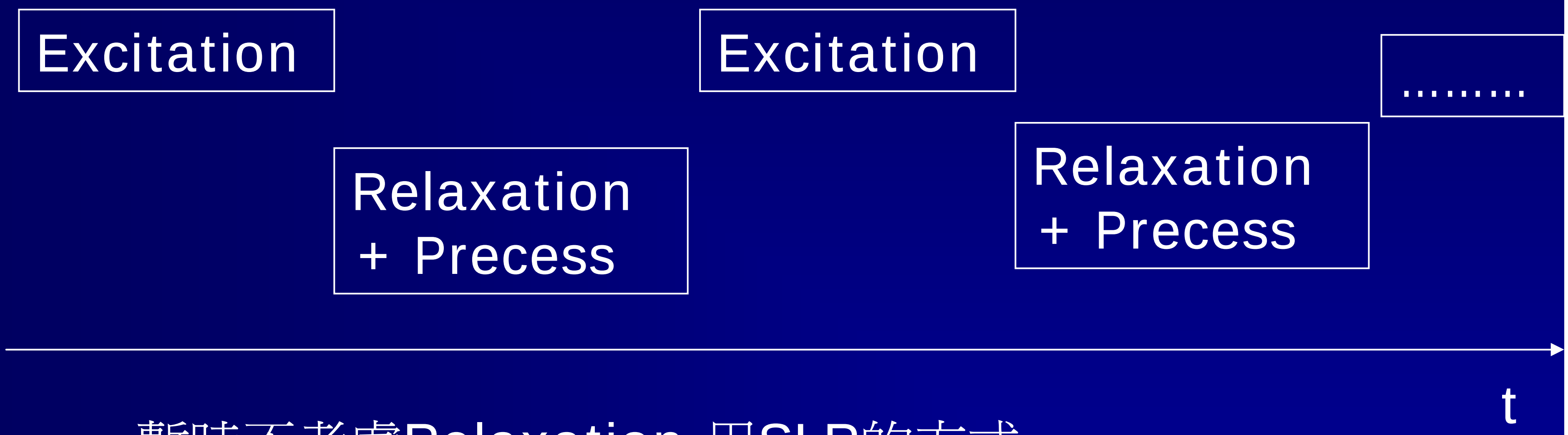


還記得最近常常看看到的圖形



Rotation with relaxation

Interesting Application



暫時不考慮Relaxation 用SLR的方式

可以約略得到Oscillating SSFP的frequency response

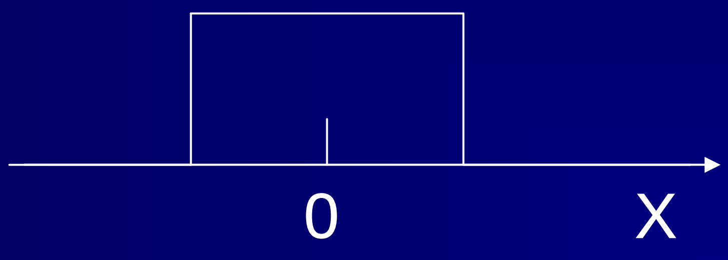
$$Q_i = \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix} \begin{bmatrix} C_i & -S_i^* \\ S_i & C_i^* \end{bmatrix}$$

Section V
SLR Pulse Design

Take it easy

純欣賞

Design A Pulse



PM FIR Design

$B_n(z)$

$A_n(z)$

$\vec{B}_{1,n}, \phi_n, \theta_n$

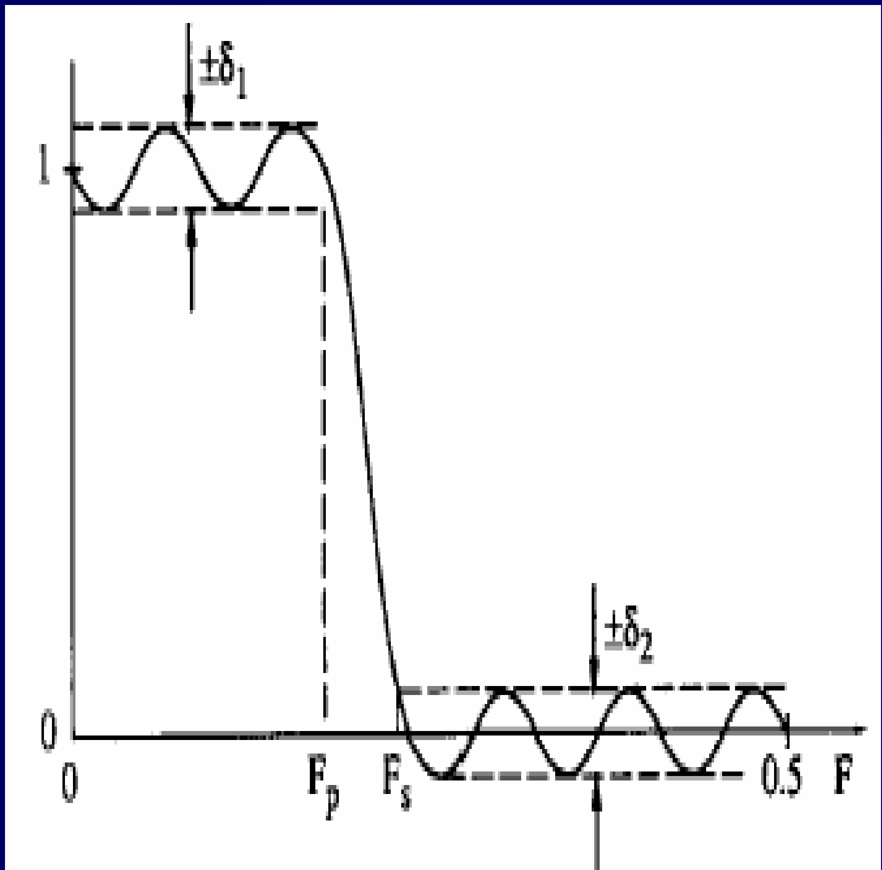
$$\begin{bmatrix} A_{i-1} \\ B_{i-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z^{1} \end{bmatrix} \begin{bmatrix} C_i & S_i^* \\ -S_i & C_i^* \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

$B_{n-1}(z)$

$A_{n-1}(z)$

N是多少就疊代幾次

PM FIR Filter Design



- Specified the parameters
  - edge of transition band
  - ripples
  - the order of the filter
  - the pulse duration
  - In-slice phase
- Trade-off between these quantities.
- 丟給Package算→$B_n(z)$
- 順便得到$A_n(z)$

Tradeoff between ripples and transition width

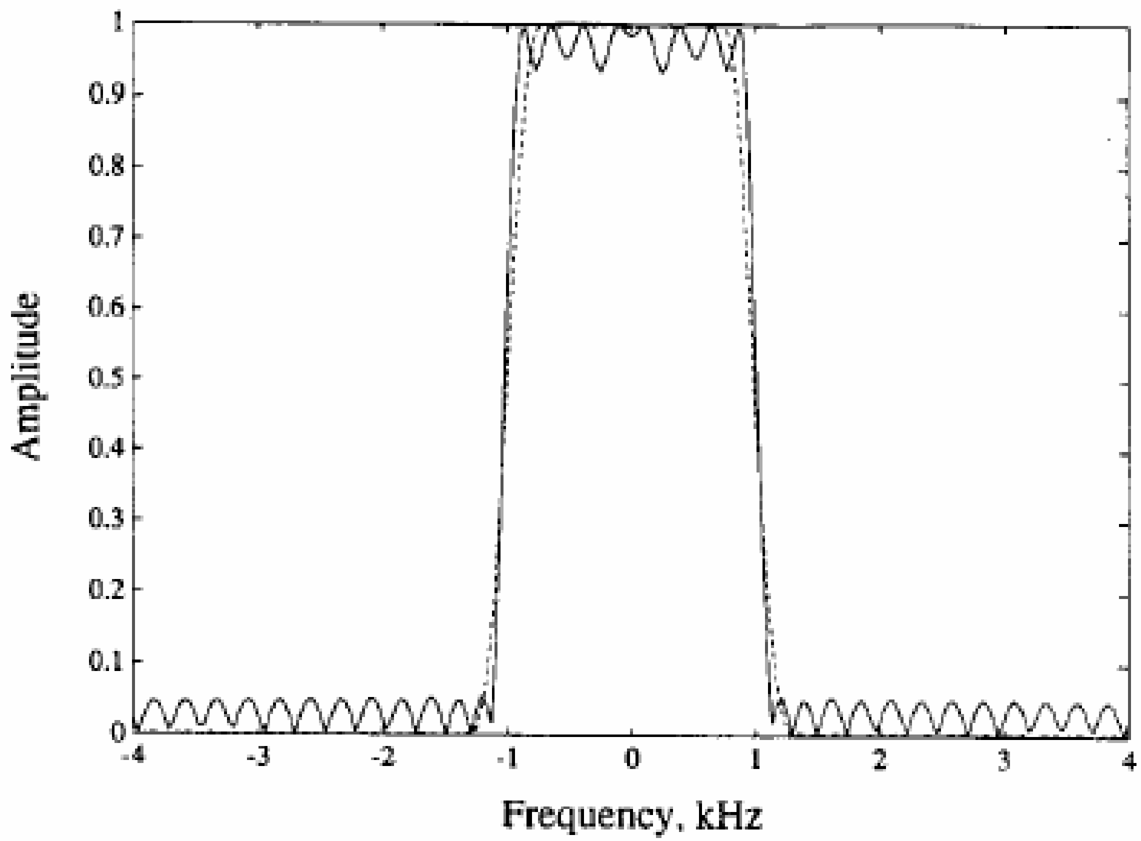


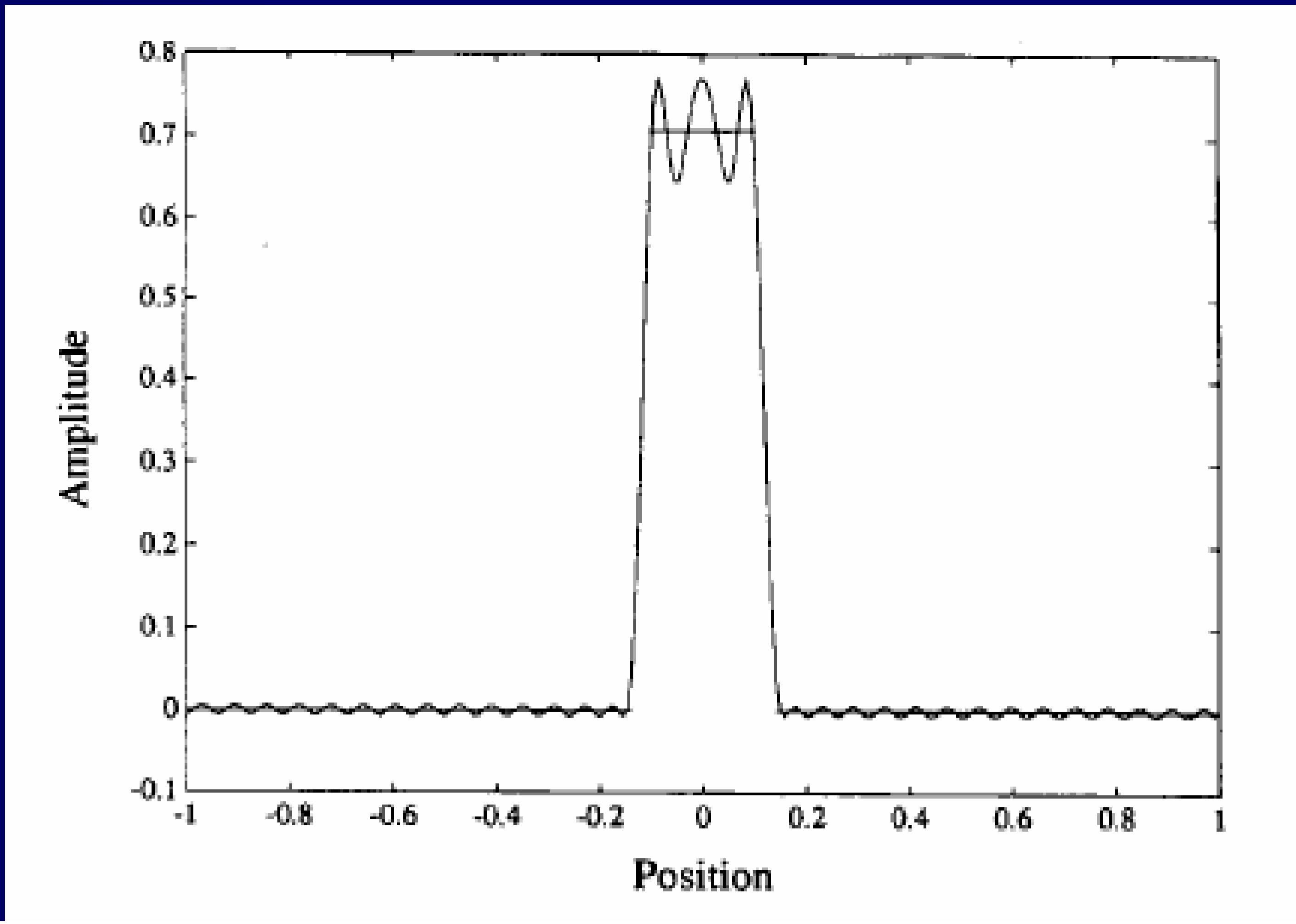
Fig. 13. Slice profiles produced by the 5% ripple (solid line) and 0.2% ripple (dashed line) SLR pulses.

In Slice Phase Selection

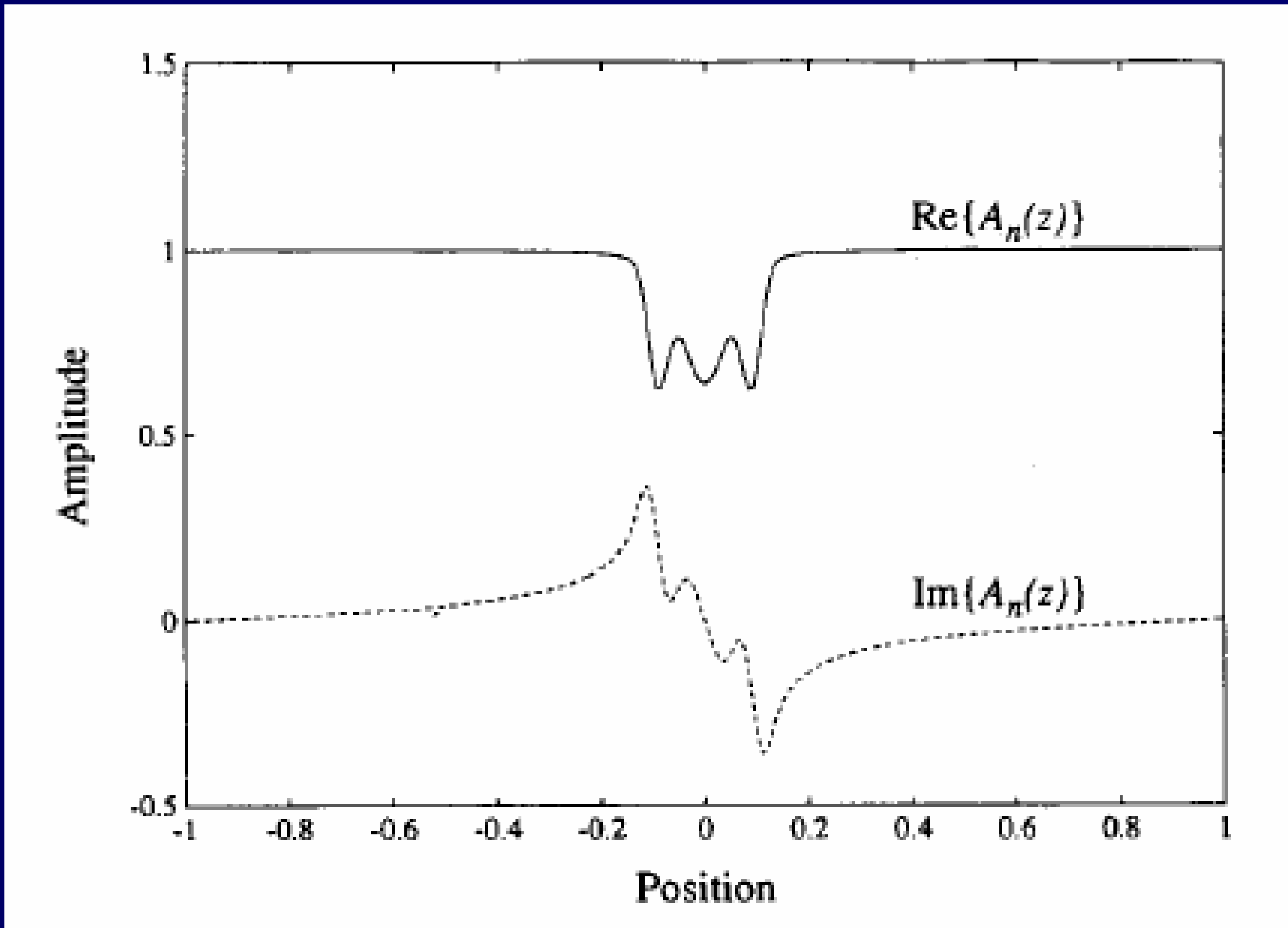
- Minimum Phase
  - Single Slice or when phase is unimportant
  - Ex: Ultra-Short TE images
- Linear Phase
  - 3D Slab excitation, Spin Echo, Phase Contrast
  - Phase is refocus by additional rephasor gradient
- Maximum Phase
  - Saturation or Inversion
  - Resulting in intra-slice dephasing as soon as possible

Linear Phase π /2 Pulse

Slice Profile and $B_n(Z)$

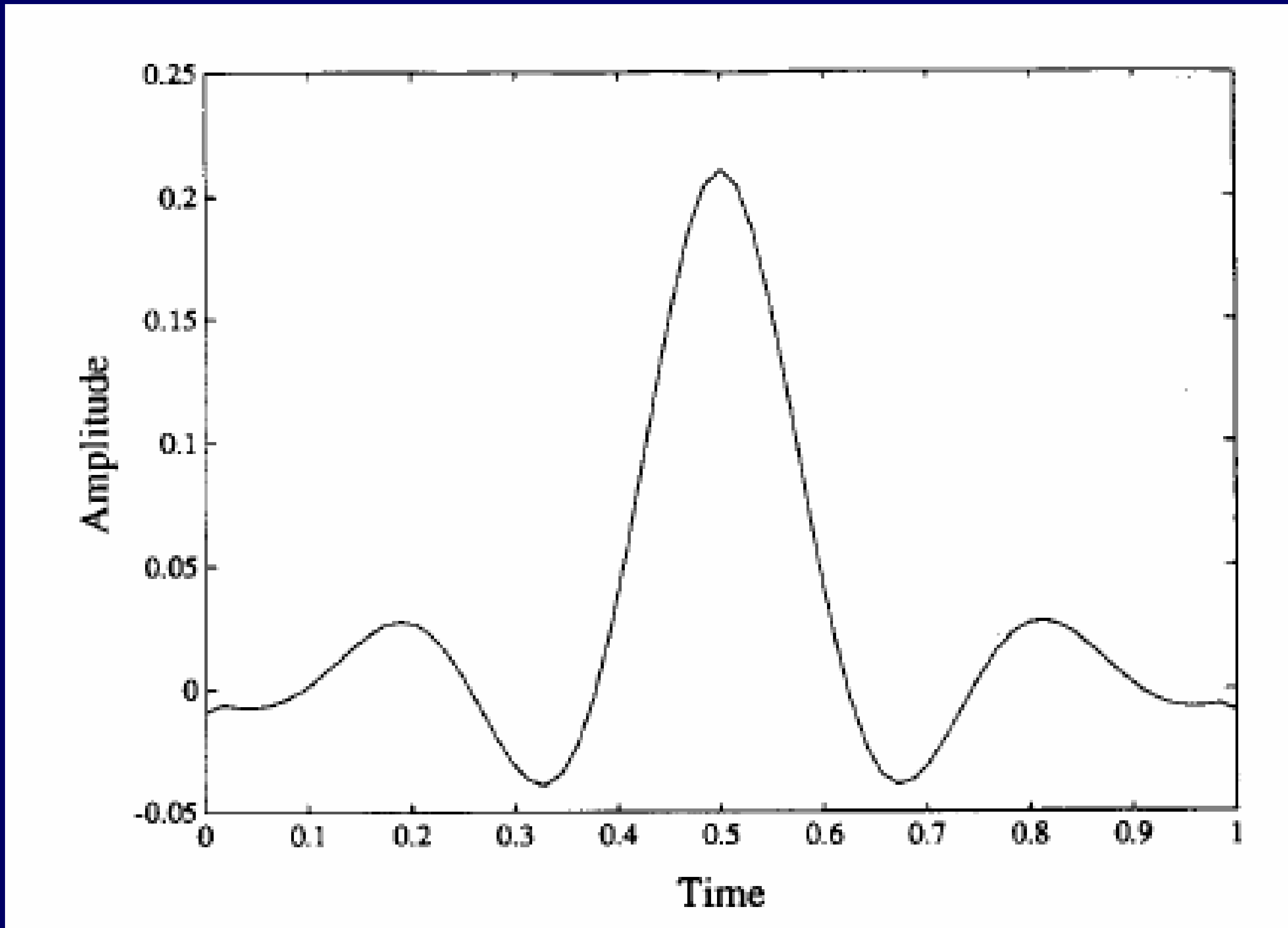


$A_n(Z)$



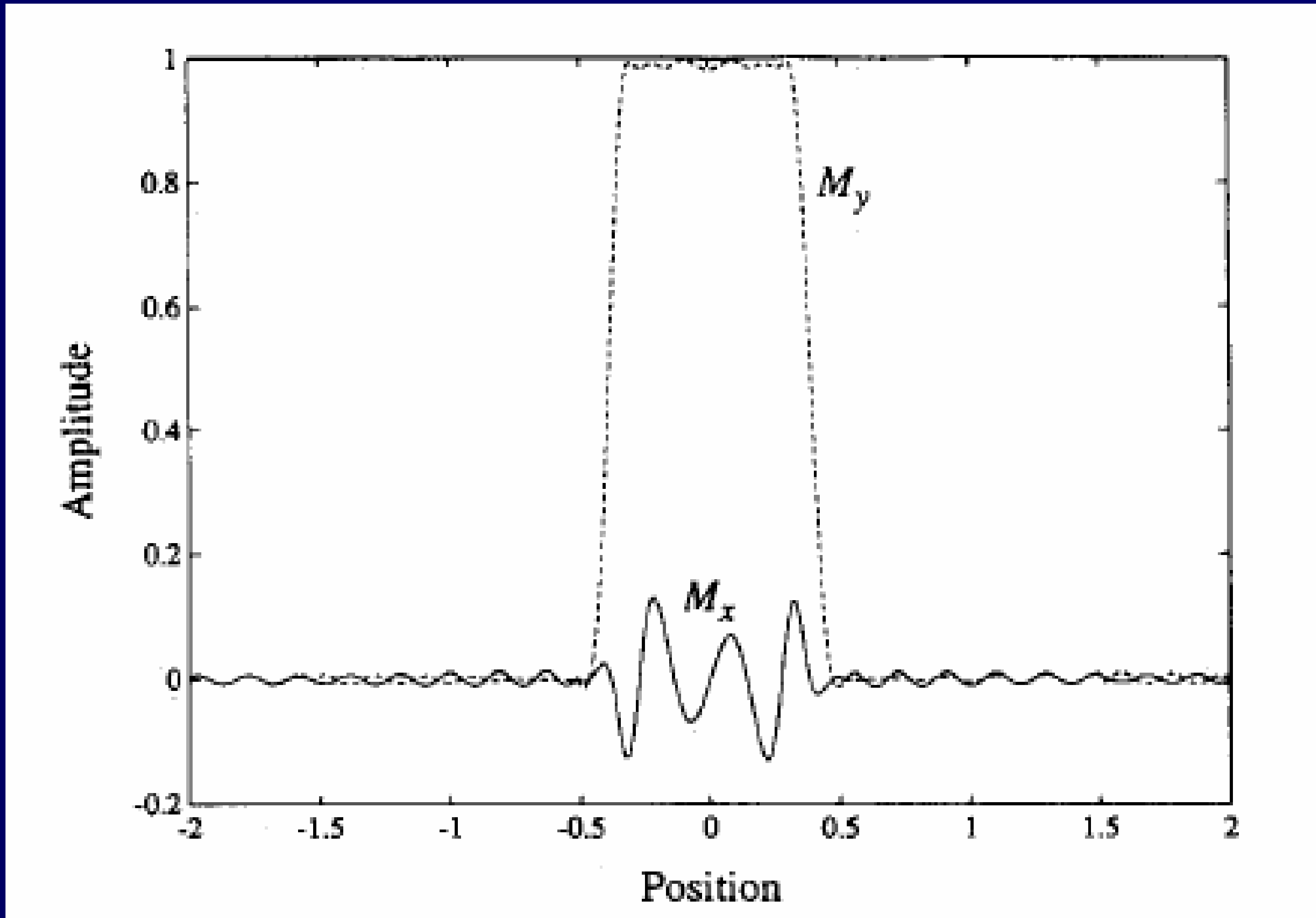
1.5
1
0.5
0
-0.5
Amplitude
Re{$A_n(z)$}
Im{$A_n(z)$}
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Position

RF Pulse

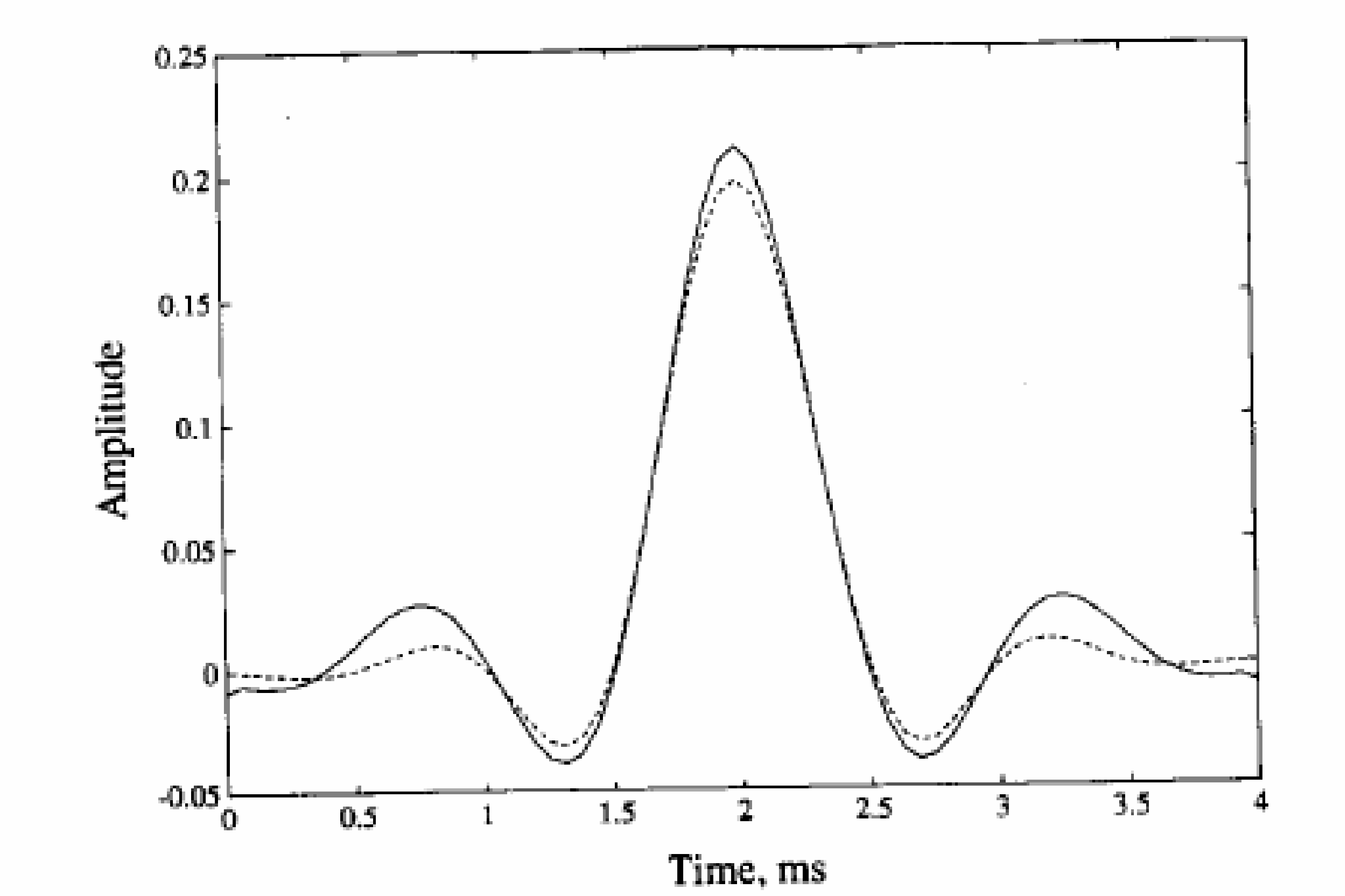


0.25
0.2
0.15
0.1
0.05
0
-0.05
Amplitude
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time

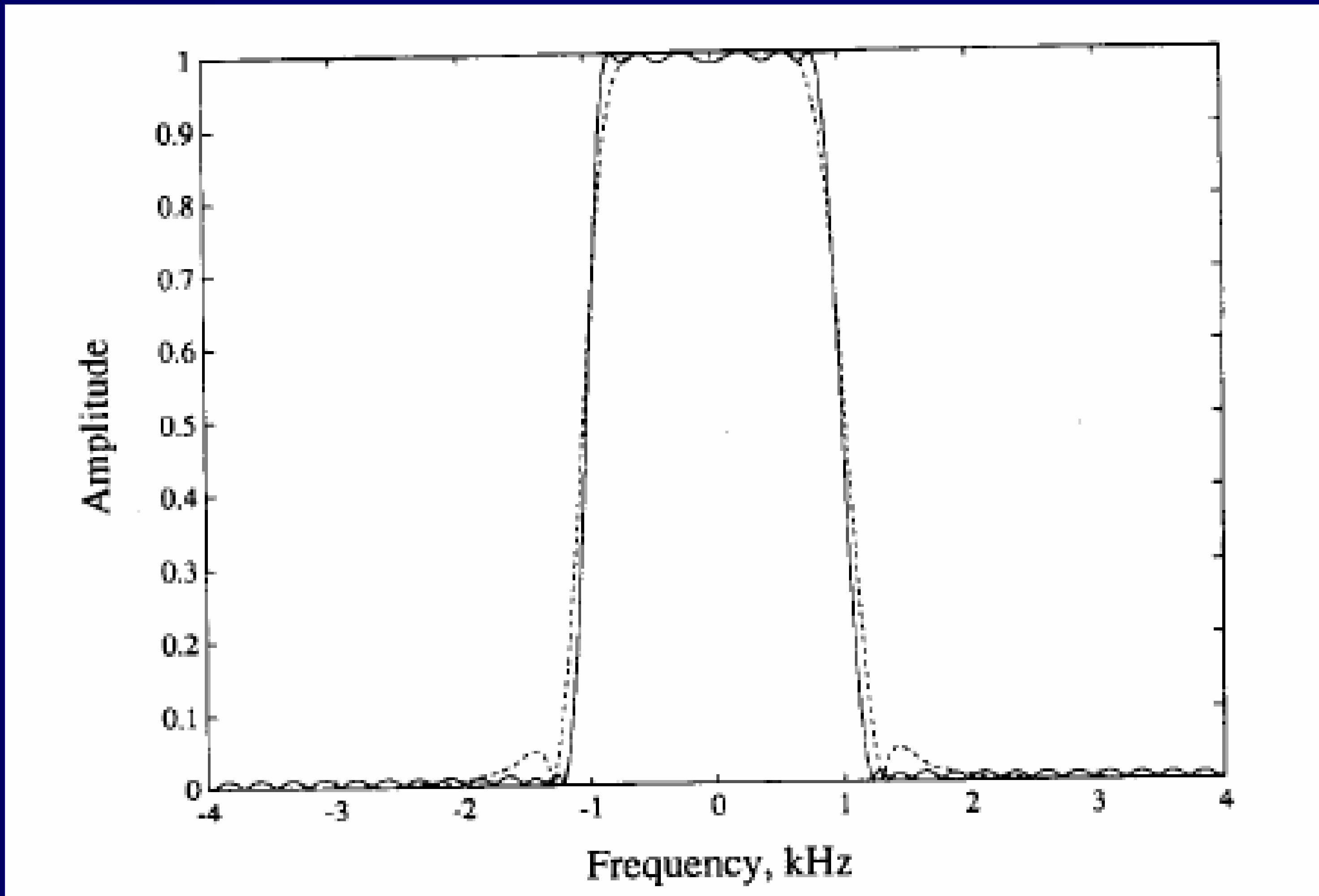
Slice Profile



Compare to Sinc RF

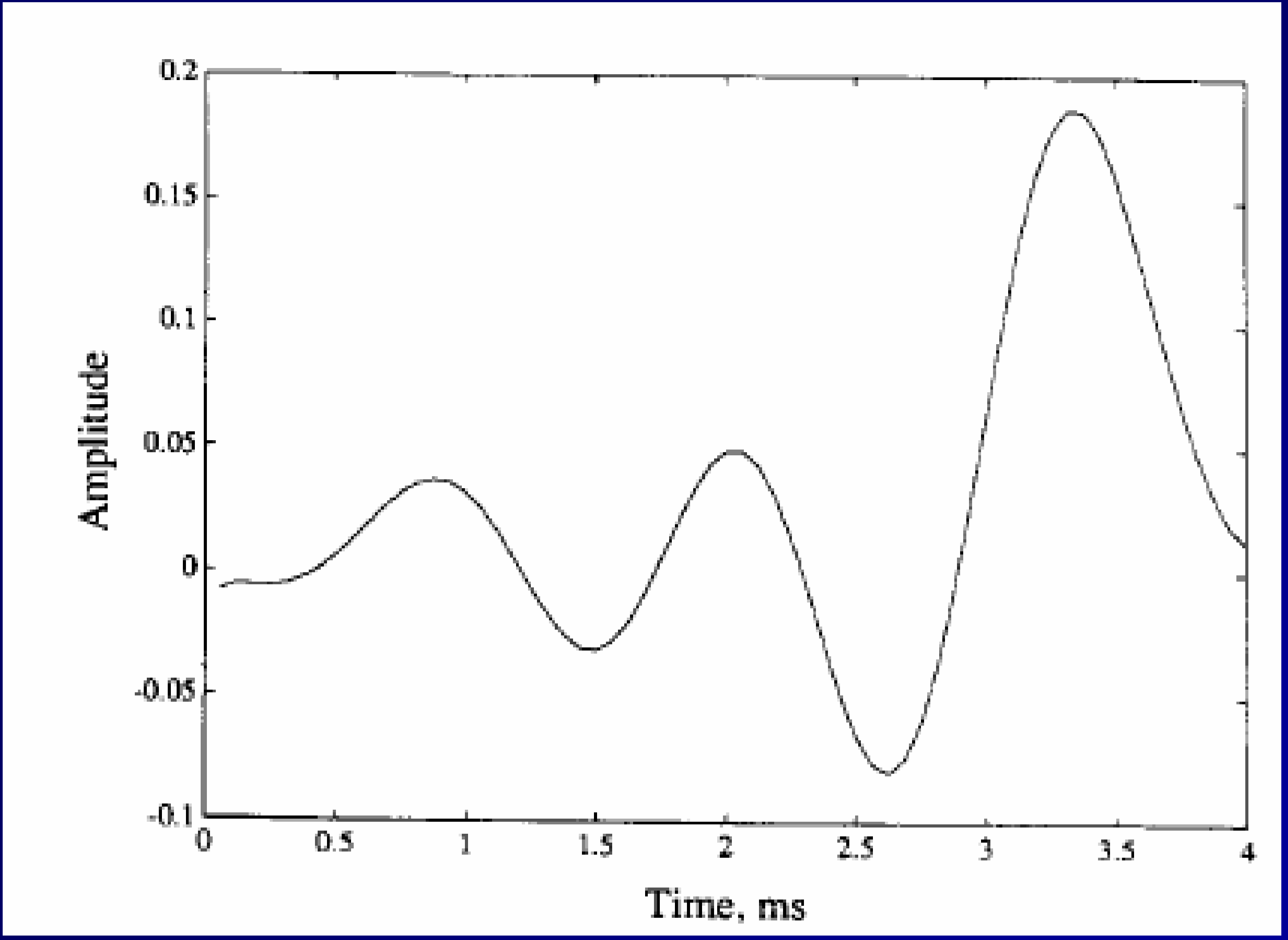


Profiles of two excitation

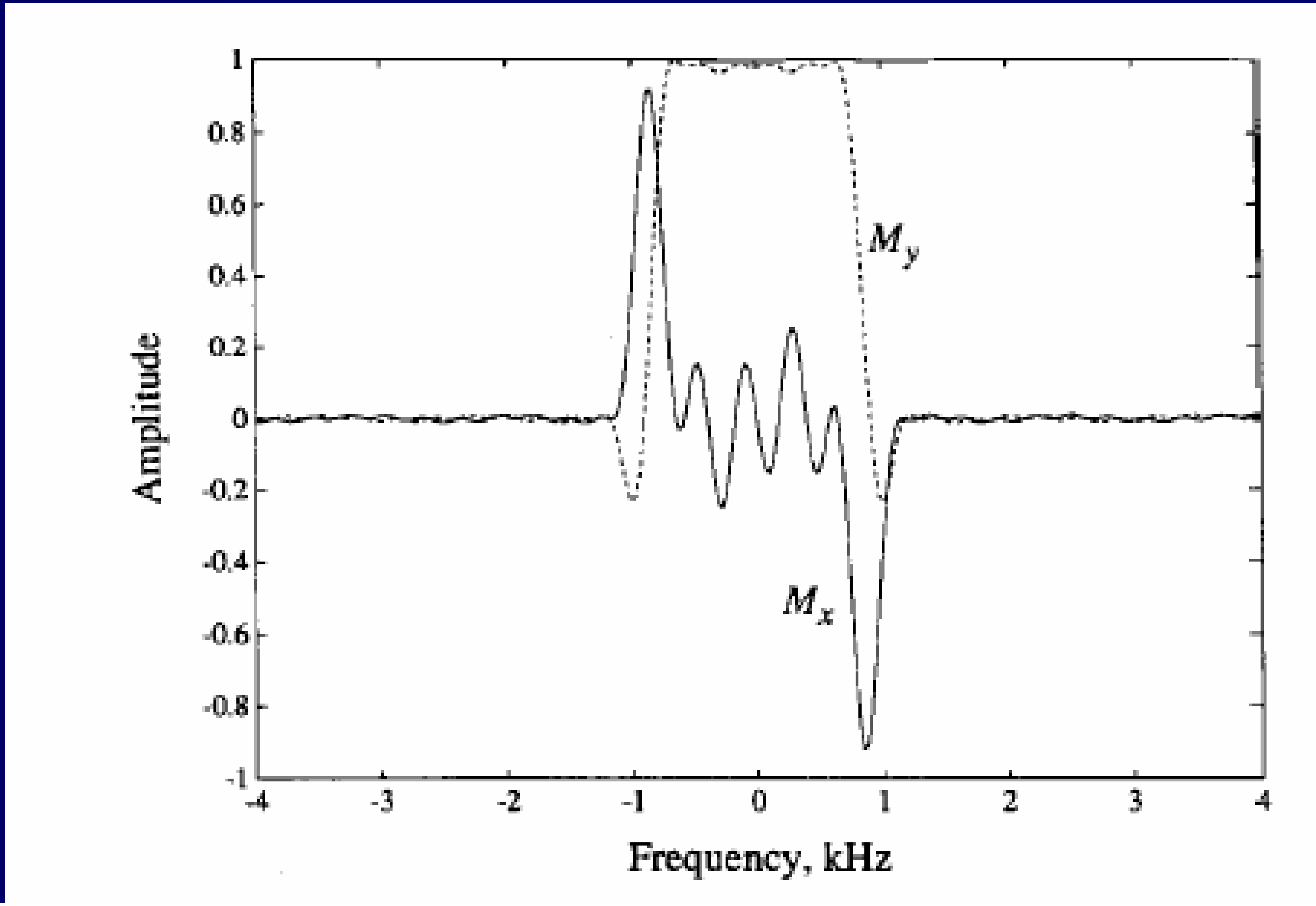


Minimum Phase π /2 Pulse

RF profile of mini-phase



Excitation Profile of Mini-Phase after refocusing



SLR Algorithm

RF Selective Excitation

$\vec{B}(t)\ \&\ G$

Hard Pulse and small angle approximation

Excitation Profile

$\vec{M}(x,t)$

Forward

Inverse

Forward

Inverse

Spin State Rotation

$A_n(Z)\ \&\ B_n(Z)$

Conclusion

- Sinc RF pulse would only suitable for small angle excitation
- SLR: a more exact solution to Bloch Equation
- SLR transform the rotation problem into a FIR design problems
- The trade-off between parameters could be analytically evaluated by filter design approach
- We could generate whatever pulses we want

Thanks for your attention

Fin

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